If $M' \subseteq M$ and $\mathcal{V} \subseteq \mathbb{C}TM'$ ($\Rightarrow$ $H \subseteq TM'$), then $[X, X']$ must be still tangent to M' for any sections X, X' of H $\Leftrightarrow$ $[Z, \bar{Z}']$ is tangent to M' for all sections Z, Z' of $\mathcal{V}$. Same is true for repeated commutators $[\cdot\cdot[X, X'], \dots X^{(l-1)}]$.

<u>Def.</u> M is of <u>finite (commutator/Hörmander) type</u> at $p \in M$ if the collection of $[\cdot\cdot[Y^1, Y^2], \dots Y^l]$, where $Y^1, Y^2, \dots, Y^l$ range over all sections of H near p, spans T_pM. The smallest l for which all such commutators span is called <u>the type</u> of M at p (if M is of finite type at p).

Thus, we have

<u>Prop.</u> M finite type at $p \Rightarrow M$ is minimal at p.

The converse is not true in general for $\mathcal{C}^\infty$-smooth CR mflds, but it is true if M is real-analytic (i.e. M is a real-analytic mfld and $\mathcal{V}$ is a real-analytic subbundle); real-analytic $=: \mathcal{C}^\omega$

To explain this, we briefly dig deeper into the notion of finite type! (not a main topic in this course). Let $\mathfrak{X}(M)$ denote the $\mathcal{C}^k$-module of vector fields (real and $\mathcal{C}^k$) on M. A Lie algebra (of vector fields) is a vector subspace ($/\mathbb{R}$) of $\mathfrak{X}(M)$ that is also closed under commutators, also known as Lie brackets,

i.e., if $X, X' \in \mathfrak{g} \Rightarrow [X, X'] \in \mathfrak{g}$.
Clearly, given any $X_1, \ldots, X_r \in \mathfrak{X}(M)$, $\exists$ a smallest Lie algebra $\mathfrak{g}$ containing $X_1, \ldots, X_r$, called the Lie algebra generated by $X_1, \ldots, X_r$.

Given the Lie algebra $\mathfrak{g}$ generated by $X_1, \ldots, X_r$, for every $p \in M$, $\mathfrak{g}(p) \subseteq T_pM$ is a vector subspace that contains $\mathbb{R}\{X_1(p), \ldots, X_r(p)\}$.

<u>Local Nakano Thm.</u> Let $X_1, \ldots, X_r$ be C^ω vector fields on M and $\mathfrak{g}$ the C^ω Lie algebra they generate. For any $p \in M$, $\exists$ open nbhd $p \in U \subseteq M$ and a unique C^ω submfld $M' \subseteq U$ s.t.

$$T_qM' = \mathfrak{g}(q), \ \forall q \in M'.$$

<u>Note</u>. Finite type at $p \Leftrightarrow \mathfrak{g}(p) = T_pM$. Thus, LNT $\Rightarrow$ Prop above.

The proof is constructive and consists of verifying that we may take M' to be the image of the exponential map

$$\mathbb{R}^m \ni (t_1, \dots, t_m) \to \exp_p(t_1 Y_1 + \dots + t_m Y_m),$$

where $Y_1, \dots, Y_m \in \mathfrak{g}$ s.t. $\mathfrak{g}(p) = \mathbb{R}\{Y_1, \dots, Y_m\}$ and the exp-map is given as follows: For any vector field X on M, let $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ be local coord's near p, $x(p) = 0$, and

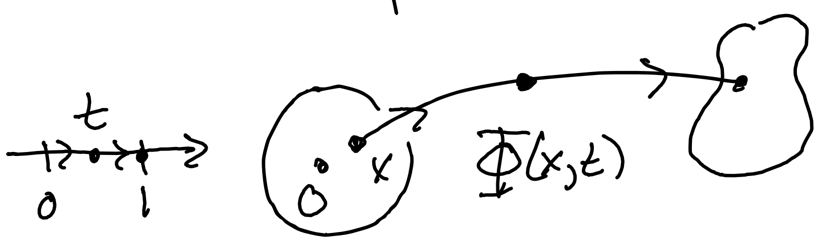
$$X = \sum_{j=1}^{N} a_j \frac{\partial}{\partial x_j}, \quad a_j \in C^k,$$

and consider the ODE initial value problem

$$\left(\frac{\partial \Phi}{\partial t} =\right) \begin{cases} \dot{\Phi}(x,t) = A(\Phi(x,t)) \\ \Phi(x,0) = x \end{cases} \qquad (*)$$

where $A = (a_1, \dots, a_N)$, $\Phi(x,t) =:$ $\Phi = (\Phi_1, \dots, \Phi_N)$.

By standard ODE theory (*) has a unique solution for suff. small t. It is easy to see that the solution to (*) for $X' = \varepsilon X$ is given by $\Phi(x, \varepsilon t) \Rightarrow$ if X is sufficiently small, then the solution $\Phi(x,t)$ exists for $t \in [0,T]$ w/ $T \geq 1$ and the map $x \to \Phi(x,t)$ is a C^k diffeomorphism from a nbhd of $x=0$ onto its image.



The exponential map $t \to \exp_x tX$ is simply $\exp_x tX = \Phi(x,t)$, where Φ is the solution of (*) corresponding to the vector field X. The details of the pf of LNT is in [BER], Sec. 3.1.